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On Complex Valued Functions with Strongly Unique Best Chebyshev Approximation*

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In contrast to the complex case, the best Chebyshev approximation with respect to a finite-dimensional Haar subspace $V \subset C(Q)$ (Q compact) is always strongly unique if all functions are real valued. However, strong uniqueness still holds for complex valued functions f with a so-called reference of maximal length. It is known that this class forms an open and dense subset in $C(Q)$ if the number of isolated points of Q does not exceed dim V. In this paper, we show that this result also holds in the space $A(Q)$ of functions, analytic in the interior of Q, if Q satisfies a certain regularity condition. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let f be a continuous function defined on a compact subset Q of the complex plane $\mathbb{C}$ and V an n-dimensional Haar subspace of $C(Q)$, where $C(Q)$ denotes the space of continuous complex valued functions on Q. We want to determine a $v \in V$ such that

$$\|f - v\| = \min_{w \in V} \|f - w\|,$$

where $\|\cdot\|$ denotes the Chebyshev norm on Q. An important role in characterizing the best approximation v to f play certain subsets of extremal points of the error curve $f - v$.

1.1. DEFINITION. Let $g \in C(Q)$ and $(z, \varphi) \in Q \times [-\pi, +\pi]$. z (as well as (z, φ)) is called *extremal point* of g if

$$g(z)\, e^{i\varphi} = \|g\|.$$

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1.2. DEFINITION. $R = \{(z_1, \varphi_1), ..., (z_m, \varphi_m)\} \subset Q \times [-\pi, +\pi]$ is called *reference* if

(i) there exist $\lambda_1, ..., \lambda_m > 0$ such that

$$\sum_{k=1}^{m} \lambda_k = 1 \qquad \text{and} \qquad \sum_{k=1}^{m} \lambda_k e^{i\varphi_k} v(z_k) = 0 \qquad \text{for all} \quad v \in V,$$

(ii) no proper subset of R satisfies (i).

A reference R is called a *reference with respect to* $f - v$, if all $(z, \varphi) \in R$ are extremal points of $f - v$.

It is known from the Kolmogoroff criterion that v is a best approximation to f iff there exists a reference R with respect to $f - v$.

Furthermore each $f \in C(Q)$ has a unique best approximation v and the length $|R|$ of a reference R with respect to $f - v$ satisfies

$$n + 1 \leqslant |R| \leqslant 2n + 1.$$

The number $|R|$ is closely related to strong uniqueness, which is very important for numerical algorithms.

1.3. DEFINITION. The best approximation v to f is *strongly unique* if there is a real number $r > 0$ such that for each $w \in V$

$$\|f - w\| \geqslant \|f - v\| + r\,\|v - w\|.$$

If the best approximation v is not strongly unique then $|R| < n + 1$ in the real case and $|R| < 2n + 1$ in the complex case [4, Thm. 2]. However, if the Haar condition is satisfied then $|R| > n$. Hence, in the real case the best approximation is always strongly unique. This is in contrast to complex approximation, where functions can be found with less than $2n$ extremal points of $f - v$ [5, Chap. 4] and so v is not strongly unique [2].

The aim of this paper is to describe the density of functions with strongly unique best approximation: Can we find a function g for fixed $f \in C(Q)$ with strongly unique best approximation such that $\|f - g\|$ is arbitrarily small?

Blatt proved in [1] that this question can be answered in the affirmative if Q has at most n isolated points. To formulate Blatt's theorem we need the following notation.

Let $f \in C(Q)$ and $v \in V$ be its best approximation. We denote by $m(f) := \min\{|R|: R \text{ reference with respect to } f - v\}$. Furthermore, for $k \in \{n + 1, ..., 2n + 1\}$ we set

$$T_k := \{f \in C(Q) : m(f) = k\}.$$

1.4. THEOREM (Blatt [1]). (i) *T_{2n+1} is an open subset of $C(Q)$.*

(ii) *T_{2n+1} is dense in $C(Q)$ iff Q has at most n isolated points.*

The terms "open" and "dense" in Theorem 1.4 refer to the topology generated by the Chebyshev norm.

However, it remained open whether an analogous result holds if only analytic functions are considered instead of the whole space $C(Q)$.

For abbreviation, let

$$A(Q) := \{f \in C(Q) : f \text{ analytic in the interior of } Q\}$$

$$H(Q) := \{f \in C(Q) : f \text{ analytic in a neighbourhood of } Q\}$$

$$P(Q) := \{f \in C(Q) : f \text{ polynomial on } Q\}.$$

The purpose of this paper is to show that $T_{2n+1} \cap P(Q)$ is dense in $A(Q)$ if Q has at most n isolated points and satisfies a certain regularity condition [5]. This regularity condition reads as follows:

1.5. DEFINITION. The compact set Q satisfies *the condition* (*), if the complement $\mathbb{C} \backslash Q$ is connected and if to each point z_0 of the boundary ∂Q of Q there exists a continuous function $\varphi : Q \backslash \{z_0\} \to \mathbb{R}$ and a constant $\gamma > 0$ such that

$$\varphi(Q \backslash \{z_0\}) \subset [-\gamma, +\gamma]$$

and

$$z - z_0 = |z - z_0| \, e^{i\varphi(z)} \qquad \text{for} \quad z \in Q \backslash \{z_0\}.$$

1.6. EXAMPLES. (a) Let Q be compact with connected complement. If for each $z_0 \in \partial Q$ there exists $\psi \in [-\pi, +\pi]$ and $\delta_0 > 0$ such that for $0 < \delta \leqslant \delta_0$

$$z_0 + \delta e^{i\psi} \in \mathbb{C} \backslash Q,$$

then (*) holds.

(b) If Q is a compact spiral around the origin, for instance

$$Q = \{e^{-t+it} : t \geqslant 0\} \cup \{0\},$$

then (*) does not hold.

2. The Main Theorem

2.1. Theorem. *Let Q comply with (*) and let $V \subset A(Q)$. Then the following statements hold:*

(i) $T_{2n+1} \cap A(Q)$ *is an open subset of* $A(Q)$.

(ii) $T_{2n+1} \cap P(Q)$ *is dense in* $A(Q)$ *if and only if* Q *has at most* n *isolated points.*

To prove Theorem 2.1 we need the following two lemmas. The first one will be shown in Section 3. The second one is part of Blatt's proof of his Main Lemma [1, pp. 163–166].

2.2. Lemma. *Let Q comply with (*), $f \in H(Q)$, $\varepsilon > 0$. Furthermore, let $m, r \in \mathbb{N}$, $r \geqslant m$, $z_0, ..., z_m \in \partial Q$, and $y_0, ..., y_m \in \mathbb{C} \setminus \{0\}$, as well as additional points $z_{m+1}, ..., z_r \in Q$. The points $z_0, ..., z_r$ are pairwise distinct.*

Then there are a function $g \in A(Q)$ and open neighbourhoods $U_0, ..., U_m$ of $z_0, ..., z_m$ such that the following properties hold:

(i) $g(z_k) = y_k$ *for* $k = 0, ..., m$.

(ii) $g(z_k) = 0$ *for* $k = m+1, ..., r$.

(iii) *For* $k = 0, ..., m$ *and* $z \in (Q \cap U_k) \setminus \{z_k\}$:

(a) $|f(z)| + |g(z)| < |f(z_k)| + |g(z_k)|$ *and*

(b) $|g(z)| < |g(z_k)|$.

(iv) *For* $z \in Q \setminus \bigcup_{k=0}^{m} U_k$: $|g(z)| < \varepsilon$.

2.3. Lemma (Blatt [1]). *Let $v \in V$ be the best approximation to $f \in C(Q)$ with*

$$R = \{(z_1, \varphi_1), ..., (z_m, \varphi_m)\}$$

as reference with respect to $f - v$, $m \leqslant 2n$. If Q has at most n isolated points then to each $\varepsilon > 0$ there exists a reference

$$\tilde{R} = \{(z_0, \tilde{\varphi}_0), ..., (z_m, \tilde{\varphi}_m)\}$$

with length $m+1$ and $z_0 \in Q \setminus \{z_1, ..., z_m\}$ such that for $j = 0, ..., m$

$$|(f - v)(z_j) - e^{-\tilde{\varphi}_j} \|f - v\| | \leqslant \varepsilon.$$

Proof of Theorem 2.1. (i) follows from Theorem 1.4(i). A proof that in (ii) the limitation of the number of isolated points is necessary can be found in [1] and in [5]. To prove the sufficiency let Q have at most n isolated points. Fix $f \in A(Q) \cap T_m$, $m \leqslant 2n$, with best approximation $v \in V$

and reference $R = \{(z_1, \psi_1), ..., (z_m, \psi_m)\}$ with respect to $f - v$. First we want to show that f can be approximated uniformly by elements of $T_{m+1} \cap A(Q)$.

Fix $\varepsilon > 0$. Since $f, v \in A(Q)$ we conclude by the maximum principle for analytic functions that $z_1, ..., z_m \in \partial Q$. Applying Lemma 2.3 we obtain a reference

$$\tilde{R} = \{(z_0, \varphi_0), ..., (z_m, \varphi_m)\}$$

with length $m + 1$ such that for $k = 0, ..., m$

$$|(f - v)(z_k) - e^{-i\varphi_k} \|f - v\|| \leqslant \frac{\varepsilon}{8}. \tag{1}$$

We observe that ∂Q has at most n isolated points because this is true for Q. Moreover, the point z_0 can be chosen as a boundary point of Q. Hence, we may apply Lemma 2.2 to $z_0, ..., z_m$.

First we show that there exists $\tilde{f} \in A(Q)$ such that

$$(\tilde{f} - v)(z_j) = e^{-i\varphi_j} \|f - v\| \qquad \text{for} \quad j = 0, ..., m, \tag{2}$$

$$|(\tilde{f} - v)(z)| < \|f - v\| \qquad \text{for} \quad z \in Q \setminus \{z_0, ..., z_m\}, \tag{3}$$

$$\|\tilde{f} - f\| \leqslant \varepsilon. \tag{4}$$

Hence $\|\tilde{f} - v\| = \|f - v\|$ and the set of extremal points of $\tilde{f} - v$ consists exactly of the points $z_0, ..., z_m$. Using (i), (ii) and (iii)(b) of Lemma 2.2, there exists a function $g_1 \in A(Q)$ such that for $j = 0, ..., m$

$$g_1(z_j) = \left(1 - \frac{\varepsilon}{4\|f - v\|}\right)(e^{-i\varphi_j} \|f - v\| - (f - v)(z_j)) \tag{5}$$

and

$$\|g_1\| = \max_{j = 0, ..., m} |g_1(z_j)|.$$

We may assume that $\varepsilon < 4\|f - v\|$. Thus due to (1)

$$\|g_1\| < \frac{\varepsilon}{8}. \tag{6}$$

Let

$$f_1 := \left(1 - \frac{\varepsilon}{4\|f - v\|}\right)(f - v) + g_1, \tag{7}$$

then $f_1 \in A(Q)$ and

$$\|f_1\| < \|f - v\| - \frac{\varepsilon}{8}, \tag{8}$$

$$|f_1(z_j)| = \|f - v\| - \frac{\varepsilon}{4}, \tag{9}$$

$$\|f_1 + v - f\| < \frac{3}{8}\varepsilon. \tag{10}$$

Property (9) is obvious, (8) and (10) are consequences of (6).

Since the complement of Q is connected, Mergelyan's Theorem [3] yields $f_2 \in P(Q)$ with

$$\|f_1 - f_2\| \leqslant \frac{\varepsilon}{16}. \tag{11}$$

Furthermore, we may assume that the following interpolation conditions hold [6, Chap. XI, Thm. 1]:

$$f_2(z_k) = f_1(z_k) \qquad \text{for} \quad k = 0, ..., m. \tag{12}$$

Now, by Lemma 2.2 we obtain $g_2 \in A(Q)$ and open neighbourhoods $U_0, ..., U_m$ of $z_0, ..., z_m$ such that for $k = 0, ..., m$

$$g_2(z_k) = \frac{\varepsilon}{4} e^{-i\varphi_k}, \tag{13}$$

$$|f_2(z)| + |g_2(z)| < |f_2(z_k)| + |g_2(z_k)| \qquad \text{for} \quad z \in (Q \cap U_k)\backslash\{z_k\}, \tag{14}$$

$$|g_2(z)| < |g_2(z_k)| \qquad \text{for} \quad z \in (Q \cap U_k)\backslash\{z_k\}, \tag{15}$$

$$|g_2(z)| \leqslant \frac{\varepsilon}{16} \qquad \text{for} \quad z \in Q \Big\backslash \bigcup_{k=0}^{m} U_k. \tag{16}$$

Consequently,

$$\|g_2\| = \frac{\varepsilon}{4}. \tag{17}$$

Let $\tilde{f} := f_2 + v + g_2$. We shall show that $\tilde{f}$ satisfies (2)–(4). First, for $j = 0, ..., m$

$$
\begin{aligned}
\tilde{f}(z_j) - v(z_j) &= f_2(z_j) + g_2(z_j) \\
&= f_1(z_j) + \frac{\varepsilon}{4} e^{-i\varphi_j} \qquad \text{(by (12) and (13))} \\
&= \left(1 - \frac{\varepsilon}{4\,\|f - v\|}\right) e^{-i\varphi_j} \|f - v\| + \frac{\varepsilon}{4} e^{-i\varphi_j} \quad \text{(by (5) and (7))} \\
&= e^{-i\varphi_j} \|f - v\|.
\end{aligned}
$$

Therefore, $\tilde{f}$ satisfies (2). To prove (3), let $z \in Q \setminus \{z_0, ..., z_m\}$ and consider the following two situations.

Case 1. There is an index $j \in \{0, ..., m\}$ with $z \in U_j$: Then (9), (12)–(14) yield

$$
\begin{aligned}
|(\tilde{f} - v)(z)| &\leqslant |f_2(z)| + |g_2(z)| \\
&< |f_2(z_j)| + |g_2(z_j)| \\
&= |f_1(z_j)| + \frac{\varepsilon}{4} \\
&= \|f - v\| - \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \|f - v\|.
\end{aligned}
$$

Case 2. $z \in Q \setminus \bigcup_{j=0}^{m} U_j$: Then

$$
\begin{aligned}
|(\tilde{f} - v)(z)| &= |f_1(z) + f_2(z) - f_1(z) + g_2(z)| \\
&\leqslant \|f_1\| + \|f_2 - f_1\| + |g_2(z)| \\
&\leqslant \|f - v\| - \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} \\
&= \|f - v\|,
\end{aligned}
$$

where the inequalities (8), (11) , and (16) were applied. Hence, (3) is true. Using (17), (10), and (11),

$$
\begin{aligned}
\|\tilde{f} - f\| &= \|\tilde{f} - v - f_2 + f_2 - f_1 + f_1 + v - f\| \\
&\leqslant \|g_2\| + \|f_2 - f_1\| + \|f_1 + v - f\| \\
&\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{16} + \frac{3}{8}\varepsilon < \varepsilon,
\end{aligned}
$$

and (4) is proved.

From (2) and (3) we conclude that $\tilde{R}$ is a reference with respect to $\tilde{f}-v$ and $\tilde{f}\in T_{m+1}$. Iterating this method, we construct $\tilde{f}\in T_{2n+1}$ with $\|\tilde{f}-f\|<\varepsilon$. Hence $A(Q)\cap T_{2n+1}$ is dense in $A(Q)$. On the other hand, T_{2n+1} is open and $P(Q)$ is dense in $A(Q)$ by the Theorem of Mergelyan. Hence, $P(Q)\cap T_{2n+1}$ is dense in $A(Q)$ and the theorem is proved.

3. Proof of Lemma 2.2

Due to condition (*) there exists for any $k\in\{0, ..., m\}$ a continuous function

$$\varphi : Q\backslash\{z_k\}\to\mathbb{R}$$

and $\gamma>1$ such that

$$z-z_k=|z-z_k|\, e^{i\varphi(z)}, \qquad |\varphi(z)|<\frac{\pi}{4}\gamma$$

for $z\in Q\backslash\{z_k\}$. Now we define on $Q\backslash\{z_k\}$ a branch of the logarithm by $\log(z-z_k):=\log(|z-z_k|)+i\varphi(z)$. Obviously, $\log(z-z_k)$ is continuous and single-valued on $Q\backslash\{z_k\}$ and analytic in the interior of this set. Furthermore, let us define

$$q_k(z):=e^{\log(z-z_k)/\gamma} \qquad \text{for} \quad z\in Q\backslash\{z_k\}$$

and

$$q_k(z_k):=0.$$

Then for $z\in Q\backslash\{z_k\}$

$$\begin{aligned} q_k(z)&=e^{\log(|z-z_k|)/\gamma}\, e^{i\varphi(z)/\gamma}\\ &=|z-z_k|^{1/\gamma}\, e^{i\psi(z)} \end{aligned}$$

with $\psi(z)\in(-\pi/4, +\pi/4)$. Since $\lim_{z\to z_k} q_k(z)=0$, we conclude $q_k\in A(Q)$. For $k=0, ..., m$ we define

$$p_k(z):=\prod_{j=0,\, j\neq k}^{m}(z-z_j)\prod_{j=m+1}^{r}(z-z_j),$$

and for $\varrho>0$

$$g_\varrho(z):=\sum_{k=0}^{m}\frac{p_k(z)\, y_k}{p_k(z)+\varrho p_k(z_k)\, q_k(z)}.$$

Next, we show that there exists a $\varrho_0 > 0$ such that for all $\varrho \geqslant \varrho_0$ each polynomial

$$p_k(z) + \varrho p_k(z_k)\, q_k(z)$$

has no zeroes in Q—hence $g_\varrho \in A(Q)$—and the statements (i)–(iv) of Lemma 2.2 hold for g_ϱ. The proof is rather lengthy. Therefore we present it in several steps and start with two remarks which can easily be shown. Then we prove two estimates for the denominator g_ϱ.

3.1. *Remark.* Let $a, b \in \mathbb{C} \setminus \{0\}$, $\alpha := \arg b - \arg a$. Then $|a| + |b| \cos \alpha \leqslant |a+b|$.

3.2. *Remark.* Let $a, c, d \in \mathbb{R}$, $a > 0$, $c > 0$, $\gamma > 1$. Let the function Φ: $[0, \infty) \to \mathbb{R}$ be defined by

$$\Phi(x) = \frac{a}{1 + cx^{1/\gamma}} + dx,$$

then there exists $\delta > 0$ such that $\Phi(x) < a$ for all $x \in (0, \delta)$.

3.3. Lemma (Local Estimate of the Denominator). *There exist pairwise disjoint neighbourhoods $V_0, ..., V_m$ of $z_0, ..., z_m$ such that for $\varrho > 0$ and $z \in Q \cap V_k$ the following inequality holds:*

$$|p_k(z) + \varrho p_k(z_k)\, q_k(z)| \geqslant |p_k(z)| + \varrho\, |p_k(z_k)|\, |q_k(z)| \cos \tfrac{3}{8}\pi.$$

Moreover, the neighbourhoods can be chosen such that

$$V_k \cap \{z_{m+1}, ..., z_r\} = \varnothing \qquad \textit{for} \quad k = 0, ..., m.$$

Proof. Let $k \in \{0, ..., m\}$. Due to $p_k(z_k) \neq 0$ we have an open neighbourhood V_k of z_k such that for $z \in V_k$

$$|\arg p_k(z_k) - \arg p_k(z)| < \frac{\pi}{8}.$$

$V_0, ..., V_m$ can be chosen to be pairwise disjoint. If $z = z_k$, then the inequality holds, since $q_k(z_k) = 0$. If $z \in Q \cap V_k$, $z \neq z_k$, we use Remark 3.1 with $a := p_k(z)$, $b := \varrho p_k(z_k)\, q_k(z)$. Then

$$|\arg(b) - \arg(a)| \leqslant |\arg(p_k(z_k)) - \arg(p_k(z))| + |\arg(q_k(z))| \leqslant \tfrac{3}{8}\pi$$

and hence

$$\cos(\arg(b) - \arg(a)) > \cos \tfrac{3}{8}\pi$$

and

$$|a+b| \geqslant |a| + |b| \cos \tfrac{3}{8}\pi.$$

3.4. Lemma (Global Estimate of the Denominator). *Let* $R>0$ *and* $\delta>0$. *Then there exists a* $\tilde{\varrho}>0$ *such that for* $\varrho \geqslant \tilde{\varrho}$ *and any* $k \in \{0, ..., m\}$

$$|p_k(z) + \varrho p_k(z_k)\, q_k(z)| \geqslant R \qquad \textit{for all} \quad z \in Q \textit{ with } |z - z_k| \geqslant \delta.$$

Proof. Set

$$\tilde{\varrho} := \frac{R + \max_{0 \leqslant j \leqslant m} \max_{z \in Q} |p_j(z)|}{(\min_{0 \leqslant j \leqslant m} |p_j(z_j)|)(\min_{0 \leqslant j \leqslant m} \min_{|z - z_j| \geqslant \delta} |q_j(z)|)}.$$

Then $0 < \tilde{\varrho} < \infty$, and for $\varrho \geqslant \tilde{\varrho}$ we have

$$\begin{aligned} &|p_k(z) + \varrho p_k(z_k)\, q_k(z)| \\ &\quad \geqslant -|p_k(z)| + \tilde{\varrho}\, |p_k(z_k)|\, |q_k(z)| \\ &\quad \geqslant -|p_k(z)| + R + \max_{z \in Q,\, j = 0, ..., m} |p_j(z)| \\ &\quad \geqslant R. \end{aligned}$$

Proof of Lemma 2.2. We prove that the function g_ϱ which was defined above satisfies the conditions of Lemma 2.2. Choose $V_0, ..., V_m$ according to Lemma 3.3, $\delta > 0$ with $\{z \in Q \mid |z - z_k| < \delta\} \subset V_k$ for $k = 0, ..., m$ and $\tilde{\varrho} = \varrho_1$ according to Lemma 3.4, where $R = 1$. Let $\varrho \geqslant \varrho_1$ and $z \in Q$.

Case 1. z lies in one of the V_k: Then by Lemma 3.3,

$$|p_k(z) + \varrho p_k(z_k)\, q_k(z)| \geqslant |p_k(z)| > 0.$$

Case 2. $z \notin \bigcup_{k=0}^{m} V_k$. Then by Lemma 3.4,

$$|p_k(z) + \varrho p_k(z_k)\, q_k(z)| \geqslant R = 1.$$

Hence in both cases the denominator is different from zero. The statements (i) and (ii) hold obviously. We show (iii) for z_0: Given an $f \in H(Q)$, $z \in V_0$, $z \neq z_0$, $\varrho \geqslant \varrho_1$. Fix $M := \max\{|y_0|, ..., |y_m|\}$. We write g instead of g_ϱ. Due to Lemma 3.3 and Lemma 3.4 we have

$$
\begin{aligned}
|g(z)| &\leqslant |y_0| \frac{|p_0(z)|}{|p_0(z)+\varrho p_0(z_0)\, q_0(z)|} \\
&\quad + |z-z_0| \sum_{k=1}^{m} \frac{\prod_{j=1, j\neq k}^{r} |z-z_j|\, |y_k|}{|p_k(z)+\varrho p_k(z_k)\, q_k(z)|} \\
&\leqslant |y_0| \frac{|p_0(z)|}{|p_0(z)| + \varrho \cos \frac{3}{8}\pi\, |p_0(z_0)|\, |q_0(z)|} \\
&\quad + |z-z_0|\, M \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{r} |z-z_j|.
\end{aligned}
$$

Using

$$
|f(z)| \leqslant |f(z_0)| + |z-z_0| \left| \frac{f(z)-f(z_0)}{z-z_0} \right|,
$$

we get

$$
\begin{aligned}
&|f(z)| + |g(z)| \\
&\quad \leqslant |f(z_0)| + |g(z_0)| \left(1 \Big/ 1 + \varrho \cos \frac{3}{8}\pi \left| \frac{p_0(z_0)}{p_0(z)} \right| |q_0(z)| \right) \\
&\qquad + |z-z_0| \left(M \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{r} |z-z_j| + \left| \frac{f(z)-f(z_0)}{z-z_0} \right| \right) \\
&\quad \leqslant |f(z_0)| + |g(z_0)| \frac{1}{1+c_0 |z-z_0|^{1/\gamma}} + d_0 |z-z_0|,
\end{aligned}
$$

where

$$
c_0 := \inf_{z \in V_0 \cap Q} \left| \frac{p_0(z_0)}{p_0(z)} \right| \varrho_1 \cos \frac{3}{8}\pi > 0
$$

and

$$
d_0 := \sup_{z \in V_0 \cap Q} \left(M \sum_{k=1}^{m} \prod_{j=1, j\neq k}^{r} |z-z_j| + \left| \frac{f(z)-f(z_0)}{z-z_0} \right| \right) < \infty.
$$

With $a = |g(z_0)|$, $c = c_0$ and $d = d_0$ we determine a $\delta_0 > 0$ (3.2) such that for $0 < |z-z_0| < \delta_0$

$$
|g(z_0)| \frac{1}{1+c_0 |z-z_0|^{1/\gamma}} + d_0 |z-z_0| < |g(z_0)|.
$$

Determining analogously $\delta_1, ..., \delta_m > 0$ and setting

$$U_k := \{z \in V_k \,|\, |z - z_k| < \delta_k\} \qquad (k = 0, ..., m)$$

we have proved (iii)(a).

Obviously, we can prove (iii)(a) for a finite number of analytic functions f, getting the U_k as intersections of open neighbourhoods of z_k. Admitting a second function $f_2 = 0$, we have shown (iii)(b). To prove (iv) we define

$$R := (m+1)\, M \max_{z \in Q,\, k = 0, ..., m} |p_k(z)| \frac{1}{\varepsilon}$$

and choose a $\delta > 0$ such that $\{z \in Q \,|z - z_k| < \delta\} \subset U_k$. Due to 3.4 there is $\varrho_2 > 0$ such that for $z \in Q \setminus \bigcup_{k=0}^m U_k$ and $\varrho \geqslant \varrho_2$

$$\begin{aligned} |g(z)| &\leqslant M \sum_{k=0}^m \frac{|p_k(z)|}{|p_k(z) + \varrho p_k(z_k)\, q_k(z)|} \\ &\leqslant M \sum_{k=0}^m \frac{|p_k(z)|}{R} \\ &\leqslant M \sum_{k=0}^m \frac{\varepsilon}{(m+1) M} \\ &= \varepsilon. \end{aligned}$$

$\varrho_0 := \max\{\varrho_1, \varrho_2\}$ complies with the statements (i) to (iv).

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